

# Enumerations of half-turn symmetric alternating-sign matrices of odd order

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## Abstract

It was shown by Kuperberg that the partition function of the square-ice model related to half-turn symmetric alternating-sign matrices of even order is the product of two similar factors. We propose a square-ice model whose states are in bijection with half-turn symmetric alternating-sign matrices of odd order. The partition function of the model is expressed via the above mentioned factors. The contributions to the partition function of the states corresponding to the alternating-sign matrices having 1 or  $-1$  as the central entry are found and the related enumerations are obtained.

## 1 Introduction

An alternating-sign matrix is a matrix with entries 1, 0, and  $-1$  such that the 1 and  $-1$  entries alternate in each column and each row and such that the first and last nonzero entries in each row and column are 1. Starting from the famous conjectures by Mills, Robbins and Rumsey [1, 2] a lot of enumeration and equinumeration results on alternating-sign matrices and their various subclasses were obtained. Most of the results were proved using bijections between matrices and states of different variants of the statistical square-ice model. For the first time such a method to solve enumeration problems was used by Kuperberg [3], see also the rich in results paper [4].

The present paper is devoted to the study of enumerations of the half-turn symmetric alternating-sign matrices of odd order on the base of the corresponding square-ice model.

In Section 2 we recall the definition of the square-ice model with the domain wall boundary conditions. The states of this model are in bijection with the alternating sign-matrices. The necessary properties of the partition function of the model are established.

In Section 3 we discuss first the square-ice model related to the half-turn symmetric alternating-sign matrices of even order proposed by Kuperberg [4]. Then a square-ice model whose states are in bijection with the half-turn symmetric alternating-sign matrices of odd order is introduced. We prove that the partition sum of this model is expressed via the partition function of the square-ice model with domain wall boundary condition and the partition function of the square-ice model related to half-turn symmetric alternating-sign matrices of even order (Theorem 1). It appears that one can separate the contributions to the partition function of the states corresponding to the alternating-sign matrices having 1 and  $-1$  as the central entry (Theorem 2).

In Section 4 we consider an important special case of the overall parameter of the model which allow to prove, in particular, the enumeration conjectures by Robbins [5] on half-turn

symmetric alternating-sign matrices of odd order. It is interesting that in this case there is a factorised determinant representation of the partition function (Theorem 3).

In section 5 we relate the enumerations of half-turn symmetric alternating-sign matrices of odd order with the enumerations of general alternating-sign matrices and half-turn symmetric alternating-sign matrices of even order. In particular, we find the explicit separate refined enumerations of the half-turn symmetric alternating-sign matrices of odd order having 1 and  $-1$  in the center of matrix.

## 2 Square-ice model with domain wall boundary

### 2.1 Definition of the model

The method used by Kuperberg to prove the alternating-sign matrix conjecture is based on the bijection between the states of the square-ice model with the domain wall boundary conditions and alternating-sign matrices. To define the state space of the square ice model we consider a subset of vertices and edges of a square grid, such that each internal vertex is tetravalent and each boundary vertex is univalent. A state of a corresponding square ice model is determined by orienting the edges in such a way that two edges enter and leave every tetravalent vertex. The domain wall boundary conditions [6] fix the orientation of the edges belonging to univalent vertices in accordance with the pattern given in Figure 1. The labels  $x_i$  and  $y_i$  are the spectral

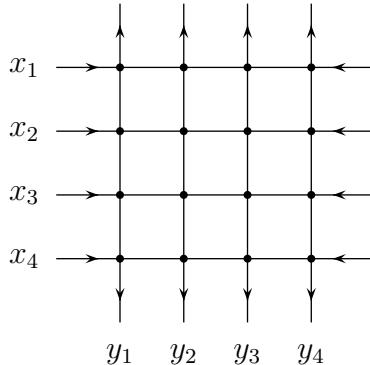


Figure 1: Square ice with a domain-wall boundary

parameters which will be used to define the partition function of the model.

If we replace each tetravalent vertex of a state of the square ice with a domain wall boundary condition by a number according to Figure 2 we will obtain an alternating-sign matrix. It is

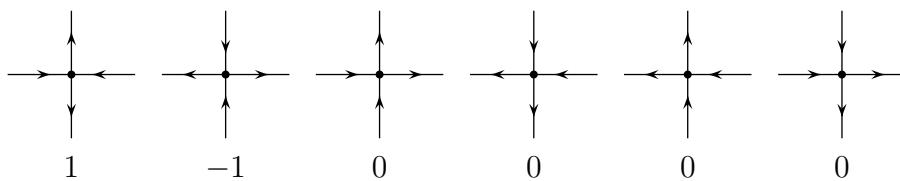


Figure 2: The correspondence between the square ice vertices and the entries of alternating-sign matrices

not difficult to check that in this way we come to the bijection between the states and the alternating-sign matrices [7, 8].

The partition function of the square ice with a domain wall boundary is the sum of the weights of all possible states of the model. The weight of a state is the product of the weights of all tetravalent vertices. To define them, we associate spectral parameters  $x_i$  with the vertical lines of the grid and spectral parameters  $y_i$  with the horizontal ones (see Figure 1). A vertex at the intersection of the line with the spectral parameter  $x_i$  and  $y_j$  is supplied with the spectral parameter  $x_i \bar{y}_j$ , where we use the notation  $\bar{x} = x^{-1}$  introduced by Kuperberg. After that we define the weights of the vertices as it is given in Figure 3, where  $a$  is a parameter common

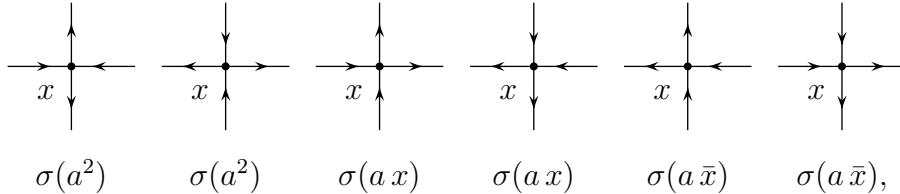


Figure 3: The weights of the vertices

for all vertices and we use the convenient abbreviations  $\sigma(x) = x - \bar{x}$  also introduced by Kuperberg.

A graph, similar to one given in Figure 1, with labeled vertices denotes the corresponding partition function. Here the summation over all possible orientations of internal edges is implied. If we have unoriented boundary edges, then the graph represents the set of the quantities corresponding to their possible orientations. It is convenient to make the formalism invariant with respect to rotations. To this end we allow a vertex label to be positioned in any quadrant related to the vertex. The value of the corresponding spectral parameter is equal to  $x_i \bar{y}_j$  if the label is placed into the quadrant swept by the line with the spectral parameter  $x_i$  when it is rotated anticlockwise to the line with the spectral parameter  $y_j$ . This rule implies that we can move a label  $x$  from one quadrant to an adjacent one, changing it to  $\bar{x}$ .

As an example we give the graph corresponding to the well-known Yang–Baxter equation, see Figure 4. This equation is satisfied if  $xyz = a$ . Note that we use the parametrization of the

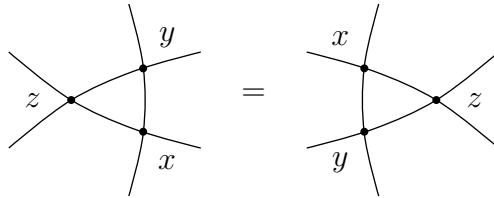


Figure 4: The Yang–Baxter equation

weights convenient for consideration of combinatorial problems proposed by Kuperberg.

## 2.2 Two lemmas on permutations

A simplest example of an alternating-sign matrix is a permutation matrix, which is defined as a matrix which can be created by rearranging the rows and columns of an identity matrix. For

any  $n \times n$  permutation matrix  $\Sigma$  one can write

$$(\Sigma)_{ij} = \delta_{is(j)},$$

where  $s$  is an appropriate unique element of the symmetric group  $S_n$ . It is clear that if we go along the column of  $\Sigma$  with the number  $j$  we meet the 1 entry at the row with the number  $s(j)$ , and if we go along the row of  $\Sigma$  with the number  $i$  we meet the 1 entry at the column with the number  $s^{-1}(i)$ , see Figure 5.

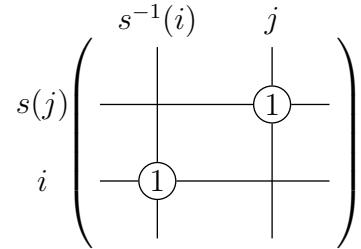


Figure 5: Positions of 1 entries in rows and columns of a permutation matrix

Recall that a pair  $(i, j)$  of integers such that  $1 \leq i < j \leq n$  is said to be an inversion of a permutation  $s \in S_n$  if  $s(i) > s(j)$ . The number of inversions of  $s$  is denoted as  $\text{inv}(s)$ . It is clear that the minimal number of inversions of an element of  $S_n$  is zero, and the maximal one is  $n(n - 1)/2$ . Introduce the generating function

$$\Phi(n; z) = \sum_{s \in S_n} z^{\text{inv}(s)} = \sum_{k=0}^{n(n-1)/2} I(n; k) z^k,$$

where  $I(n; k)$  is the number of the elements of  $S_n$  with  $k$  inversions. The following result is well known.

**Lemma 1** *For any positive integer  $n$  one has*

$$\Phi(n; z) = (1 + z)(1 + z + z^2) \cdots (1 + z + \cdots + z^{n-1}). \quad (1)$$

*Proof.* The very form of the statement of the lemma suggests to prove it by induction. Represent an element  $s$  of  $S_n$  as the word  $s(1)s(2)\dots s(n)$ . The words representing the elements of  $S_{n+1}$  can be created by inserting the letter ' $n + 1$ ' into the words representing the elements of  $S_n$ . As the result we have  $n + 1$  possibilities. First, let the letter ' $n + 1$ ' is at the last position of the resulting word. The length of the obtained element of  $S_{n+1}$  is equal to the length of the initial element of  $S_n$ . This possibility gives a contribution to  $\Phi(n + 1; z)$  coinciding with  $\Phi(n; z)$ . Let now the letter ' $n + 1$ ' is at the next to last position. The length of the obtained permutation is one more than the length of the initial one. This gives a contribution  $\Phi(n; z)z$ . Exhausting all the possibilities we obtain the equality

$$\Phi(n + 1; z) = \Phi(n; z)(1 + z + \cdots + z^n).$$

Taking into account that  $\Phi(1; z) = 1$ , we come to the statement of the lemma.  $\square$

**Lemma 2** For a state of square ice corresponding to the permutation matrix associated with a permutation  $s \in S_n$ , the numbers of the vertices of third type and of forth type are equal to  $\text{inv}(s)$  and the numbers of the vertices of fifth type and of sixth type are equal to  $n(n - 1)/2 - \text{inv}(s)$ .

*Proof.* Consider a state of square ice corresponding to a permutation matrix  $\Sigma$  associated with a permutation  $s \in S_n$ . Note that if we walk along a row or a column of the picture representing the state under consideration, then passing a vertex of the first or the second type we change the orientation of the edge. Taking into account that the vertices of the first or the second type are situated where one finds entry 1 in the corresponding permutation matrix one can make the following conclusions. A 0 entry in the permutation matrix corresponds to a vertex of third type, if a 1 entry is below it and a 1 entry is from the right of it; a 0 entry in the permutation matrix corresponds to a vertex of forth type, if a 1 entry is above it and a 1 entry is from the left of it, see Figure 6. Similarly, a 0 entry in the permutation matrix corresponds to a vertex of

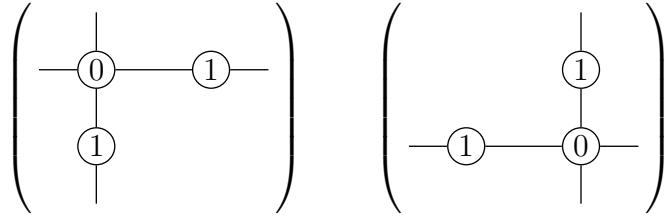


Figure 6: The disposition of the vertices of third and forth types

fifth type, if a 1 entry is below it and a 1 entry is from the left of it; a 0 entry in the permutation matrix corresponds to a vertex of sixth type, if a 1 entry is above it and a 1 entry is from the right of it, see Figure 7.

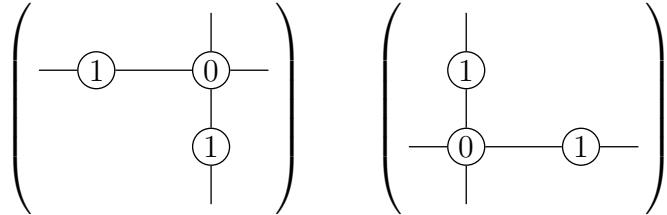


Figure 7: The disposition of the vertices of fifth and sixth types

Let now  $(i, j)$  be an inversion of  $s$ . Consider the item at the intersection of the row with the number  $s(i)$  and the column with the number  $j$  of the matrix  $\Sigma$ . Since  $(i, j)$  is an inversion, we have the situation depicted in Figure 8. It is clear that the state of square ice have at the considered position a vertex of forth type. It is not difficult to get convinced that we have a bijection between the inversions of  $s$  and the vertices of the forth type of the state. Hence, the number of the vertices of forth type is equal to  $\text{inv}(s)$ . Analysing Figure 6, one sees that we have equal numbers of the vertices of the third type and of the forth type.

Further, Figure 6 shows that we have equal numbers of the vertices of the fifth type and of the sixth type. Taking into account that the total number of the vertices of the first type and of the second type is  $n$ , we see that the last statement of the lemma is true.  $\square$

$$s^{-1}(s(i)) = i \quad j$$

$$s(j) \left( \begin{array}{c} | \\ - - - \\ | \end{array} \right)$$

$$s(i) \left( \begin{array}{c} | \\ - - - \\ | \end{array} \right)$$

Figure 8: The correspondence between the inversions and the vertices of the forth type

### 2.3 Leading terms of the partition function

Denote the partition function of the square-ice model with a domain wall boundary condition by  $Z(n; \mathbf{x}, \mathbf{y})$ . Here  $n$  is the size of the square ice,  $\mathbf{x}$  and  $\mathbf{y}$  are vectors constructed from the spectral parameters,

$$\mathbf{x} = (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_n).$$

It is clear that the partition function  $Z(n; \mathbf{x}, \mathbf{y})$  is a Laurent polynomial in the variables  $x_i$  and  $y_i$ . Define the modified partition function

$$\tilde{Z}(n; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^n x_i^{n-1} y_i^{n-1} \right] Z(n; \mathbf{x}, \mathbf{y}).$$

The next lemma describes some properties of the modified partition function which will be needed below.

**Lemma 3** *The modified partition function  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  has the following properties.*

- (a) *The function  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  is symmetric separately in the variables  $x_1, \dots, x_n$  and in the variables  $y_1, \dots, y_n$ .*
- (b) *The function  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  is a homogeneous polynomial in the variables  $x_i$  and  $y_i$  of total degree  $2n(n-1)$ . For each fixed  $i = 1, \dots, n$  it is a polynomial in  $x_i^2$  of degree  $n-1$  and a polynomial in  $y_i^2$  of degree  $n-1$ .*
- (c) *If  $y_n = ax_n$ , then*

$$\tilde{Z}(n; \mathbf{x}, \mathbf{y}) = \sigma(a^2) \prod_{i=1}^{n-1} [(ay_i^2 - \bar{a}x_n^2)(ay_n^2 - \bar{a}x_i^2)] \tilde{Z}(n-1; \mathbf{x} \setminus x_n, \mathbf{y} \setminus y_n), \quad (2)$$

where  $\mathbf{x} \setminus x_n = (x_1, \dots, x_{n-1})$  and  $\mathbf{y} \setminus y_n = (y_1, \dots, y_{n-1})$ .

*Proof.* Statement (a) of the lemma follows from the symmetricity of the partition function  $Z(n; \mathbf{x}, \mathbf{y})$ . Statement (b) is actually evident. The recursive relation (2) follows from the corresponding recursive relation for the partition function  $Z(n; \mathbf{x}, \mathbf{y})$  which states that if  $y_n = ax_n$ , then

$$Z(n; \mathbf{x}, \mathbf{y}) = \sigma(a^2) \prod_{i=1}^{n-1} [\sigma(a \bar{x}_n y_i) \sigma(a \bar{x}_i y_n)] Z(n-1; \mathbf{x} \setminus x_n, \mathbf{y} \setminus y_n). \quad (3)$$

The proof of the mentioned properties of the partition function  $Z(n; \mathbf{x}, \mathbf{y})$  can be found, for example, in paper [4].  $\square$

In calculating the modified partition function  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  it is convenient to multiply the weight of a vertex at the intersection of the horizontal line with the spectral parameter  $x_i$  and the vertical line with the spectral parameter  $y_j$  corresponding to a 0 entry of the alternating-sign matrix by  $x_i y_j$ . Here the factor  $x_i y_j$  is extracted from the factor entering the definition of the modified partition function. Hence, we assume that a 0 entry corresponding to a vertex of third or forth type gives the contribution  $a x_i^2 - \bar{a} y_j^2$  to the corresponding term of the modified partition function, and a 0 entry corresponding to a vertex of fifth or sixth type gives the contribution  $a y_j^2 - \bar{a} x_i^2$ .

Singling out from  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  the term of maximal degree in the variables  $x_i$ , we write

$$\tilde{Z}(n; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^n x_i^{2(n-1)} \right] C(n) + \dots$$

**Lemma 4** *The coefficient  $C(n)$  is given by the formula*

$$C(n) = \prod_{i=1}^n \sigma(a^{2i}). \quad (4)$$

*Proof.* One can easily get convinced that to find the term of maximal degree in the variables  $x_i$  one should take only the states of the square ice which have only one 1 entry in each column of the corresponding alternating-sign matrices. Only in this case all the parameters  $y_i$  from the factor entering the definition of the modified partition function can be absorbed into weights of vertices. The alternating-sign matrices which have only one 1 entry in each column are the permutation matrices. Therefore, from Lemma 2 it follows that the contribution to the term of maximal degree in the variables  $x_i$  of the state corresponding to a permutation  $s$  is

$$\left[ \prod_{i=1}^n x_i^{2(n-1)} \right] \sigma^n(a^2) a^{2 \text{inv}(s)} (-\bar{a})^{n(n-1)-2 \text{inv}(s)} = \left[ \prod_{i=1}^n x_i^{2(n-1)} \right] \sigma^n(a^2) a^{-n(n-1)} a^{4 \text{inv}(s)}.$$

Hence, one obtains

$$C(n) = \sigma^n(a^2) a^{-n(n-1)} \sum_{s \in S_n} a^{4 \text{inv}(s)}.$$

Taking into account equality (1), after some elementary transformations we come to relation (4).  $\square$

Let us go further and consider the terms of  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  which have the maximal total degree in all the variables  $x_i$  except the variable  $x_n$ , and do not contain all the variables  $y_i$  except the variable  $y_n$ . Singling out these term, we write

$$\tilde{Z}(n; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^{n-1} x_i^{2(n-1)} \right] S(n; x_n, y_n) + \dots$$

**Lemma 5** *The polynomial  $S(n; x_n, y_n)$  has the form*

$$S(n; x_n, y_n) = \left[ \prod_{i=1}^{n-1} \sigma(a^{2i}) \right] (\sigma(a^{2n}) x_n^{2(n-1)} - \sigma(a^{2(n-1)}) x_n^{2(n-2)} y_n^2). \quad (5)$$

*Proof.* Since we are looking for the term which do not contain all the variable  $y_i$  except the variable  $y_n$ , we can consider only the states which have only one 1 entry in each of the first  $n - 1$  columns. By the definition of alternating-sign matrix one has only one 1 entry in the last column too. The first  $n - 1$  rows of such a matrix give the required degree in the variables  $x_i$ . The weight of a vertex in the last row is either  $ay_j^2 - \bar{a}x_n^2$  or  $ax_n^2 - \bar{a}y_j^2$ . One has only one entry 1 in this row. If this entry is in the last column, we have the degree  $2(n - 1)$  in the variable  $x_n$  and zero degree in the variable  $x_n$ . If the entry 1 is not in the last column we obtain either a contribution proportional  $x_n^{2(n-1)}$  or a contribution proportional to  $x_n^{2(n-2)}y_n^2$  depending on which term we take from the weight of the last vertex in the row equal to  $ax_n - \bar{a}y_n$ . Thus, we conclude that the polynomial  $S(n; x_n, y_n)$  is of the form

$$S(n; x_n, y_n) = C(n)x_n^{2(n-1)} + D(n)x_n^{2(n-2)}y_n^2.$$

From recursive relation (2) one obtains

$$C(n) + a^2D(n) = \sigma(a^2)a^{-2(n-1)}C(n-1).$$

Taking into account equality (4), one comes to the relation

$$D(n) = - \left[ \prod_{i=1}^{n-1} \sigma(a^{2i}) \right] \sigma(a^{2(n-1)})$$

which implies equality (5).  $\square$

### 3 Square-ice models related to half-turn symmetric alternating-sign matrices

#### 3.1 Half-turn symmetric alternating-sign matrices

We say that an alternating-sign matrix  $A$  is half-turn symmetric if

$$(A)_{n-1-i, n-1-j} = (A)_{i,j}.$$

The  $n \times n$  permutation matrix  $\Sigma$  associated with a permutation  $s \in S_n$  is half-turn symmetric if and only if

$$s(n-1-i) = n-1-s(i), \quad i = 1, \dots, n.$$

We denote the set of all permutations associated with  $n \times n$  half-turn symmetric permutation matrices by  $S_n^{\text{HT}}$  and introduce the generating function

$$\Phi_{\text{HT}}(n; z) = \sum_{s \in S_n^{\text{HT}}} z^{\text{inv}(s)} = \sum_{k=0}^{n(n-1)/2} I_{\text{HT}}(n; k)z^k,$$

where  $I_{\text{HT}}(n; k)$  is the number of the elements of  $S_n^{\text{HT}}$  with  $k$  inversions.

**Lemma 6** *The equalities*

$$\Phi_{\text{HT}}(2m+1; z) = \left[ \prod_{i=1}^m (1 + z^{2i+1}) \right] \Phi(m; z^2), \quad (6)$$

$$\Phi_{\text{HT}}(2m; z) = \left[ \prod_{i=1}^m (1 + z^{2i-1}) \right] \Phi(m; z^2) \quad (7)$$

are valid.

*Proof.* Equality (6) is valid for  $m = 0$ , suppose that it is valid for some  $m = k > 0$ . Let  $s$  be an arbitrary element of  $S_{2k+1}^{\text{HT}}$ . Identify it with the permutation  $s'$  of the alphabet  $\{2, 3, \dots, 2k+2\}$  defined as

$$s'(i) = s(i-1) + 1$$

and represent  $s$  as the word  $s'(2)s'(3)\dots s'(2k+2)$ . If we insert the letters ‘1’ and ‘ $2k+3$ ’ into this word in the corresponding symmetric way we obtain a word representing an element of  $S_{2k+3}^{\text{HT}}$ . It is clear that in such a way we obtain all elements of  $S_{2k+3}^{\text{HT}}$  and each element is obtained only once. If we insert the letter ‘1’ before the word and the letter ‘ $2k+3$ ’ after it, we obtain an element of  $S_{2k+3}^{\text{HT}}$  which have the same number of inversions as the initial element of  $S_{2k+1}^{\text{HT}}$ . This gives the contribution equal to  $\Phi_{\text{HT}}(2k+1; z)$  to the generating function  $\Phi_{\text{HT}}(2k+3; z)$ . If we insert the letter ‘1’ into the second position and the letter ‘ $2k+3$ ’ into the next to last position we obtain an element of  $S_{2k+3}^{\text{HT}}$  which have the number inversions greater by two as the initial element of  $S_{2k+1}^{\text{HT}}$ . This gives a contribution equal to  $\Phi_{\text{HT}}(2k+1; z)z^2$ . Continuing this procedure we exhaust all elements of  $S_{2k+3}^{\text{HT}}$ . Note that when we pass through the middle of the word the number of inversions increases by three. Thus, we have

$$\begin{aligned}\Phi_{\text{HT}}(2k+3; z) &= \Phi_{\text{HT}}(2k+1; z)(1 + z^2 + \dots z^{2k} + z^{2k+3} + \dots + z^{4k+3}) \\ &= \Phi_{\text{HT}}(2k+1; z)(1 + z^{2k+3})(1 + z^2 + \dots + z^{2k}).\end{aligned}$$

This equality implies that relation (6) is valid for  $m = k + 1$ . Hence, it is valid for any  $m \geq 0$ . Equality (7) can be proved in the same way.  $\square$

### 3.2 Square-ice model for matrices of even order

A method for constructing square-ice models corresponding to alternating-sign matrices with some symmetry was proposed by Kuperberg [4]. In this method one actually considers a subset of the vertices of the state corresponding to the full alternating-sign matrix which uniquely determines it, and specifies the spectral parameters in an appropriate convenient way. Kuperberg considered a square-ice model corresponding to half-turn symmetric alternating-sign matrices of even order. The structure of the state pattern and the specification of the spectral parameters for this model can be understood from an example given in Figure 9.<sup>1</sup> One has the following

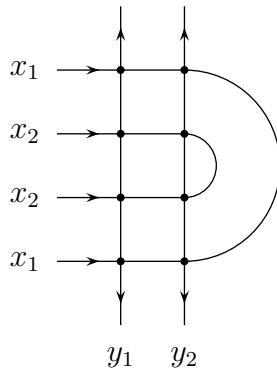


Figure 9: Square ice with a half-turn symmetric boundary of even size

evident analogue of Lemma 2.

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<sup>1</sup>Actually Kuperberg introduced two variants of the model differing by specification of the spectral parameters. For our purposes it suffices to consider only one variant.

**Lemma 7** For a state of square ice with half-turn symmetric boundary of even size corresponding to the permutation matrix associated with a permutation  $s \in S_{2m}^{\text{HT}}$ , the total number of the vertices of third type and of forth type is equal to  $\text{inv}(s)$  and the total number of the vertices of fifth type and of sixth type is equal to  $m(2m - 1) - \text{inv}(s)$ .

We will denote the partition function of square ice with half-turn symmetric boundary by  $Z_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$ .<sup>2</sup> This function is a Laurent polynomial in the variables  $x_i$  and  $y_i$ . Introduce the modified partition function

$$\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^m x_i^{2m-1} y_i^{2m-1} \right] Z_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}).$$

**Lemma 8** The modified partition function  $\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$  has the following properties.

- (a) The function  $\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$  is symmetric separately in the variables  $x_1, \dots, x_m$  and in the variables  $y_1, \dots, y_m$ .
- (b) The function  $\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$  is a homogeneous polynomial in the variables  $x_i$  and  $y_i$  of total degree  $2m(2m - 1)$ . For each fixed  $i = 1, \dots, m$  it is a polynomial in  $x_i^2$  of degree  $2m - 1$  and a polynomial in  $y_i^2$  of degree  $2m - 1$ .
- (c) If  $y_m = ax_m$ , then

$$\begin{aligned} \tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}) &= \sigma^2(a^2) x_m y_m \\ &\quad \prod_{i=1}^{m-1} [(ay_i^2 - \bar{a}x_m^2)^2 (ay_m^2 - \bar{a}x_i^2)^2] \tilde{Z}_{\text{HT}}(2m - 2; \mathbf{x} \setminus x_m, \mathbf{y} \setminus y_m). \end{aligned}$$

*Proof.* The stated properties of the modified partition function  $\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$  follow from the corresponding properties of the partition function  $Z_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$ , see [4].  $\square$

Singling out from  $\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$  the term of maximal degree in the variables  $x_i$ , we write

$$\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^m x_i^{2(2m-1)} \right] C_{\text{HT}}(2m) + \dots$$

**Lemma 9** The coefficient  $C_{\text{HT}}(2m)$  is given by the formula

$$C_{\text{HT}}(2m) = \prod_{i=1}^{2m} \sigma(a^i). \tag{8}$$

*Proof.* It is clear that only the states corresponding to permutation matrices contribute to the term of maximal degree in the variables  $x_i$ . Using the same reasonings as in the proof of Lemma 4 and taking into account Lemma 7, we see that the contribution of a state corresponding to a permutation  $s$  is

$$\begin{aligned} &\left[ \prod_{i=1}^m x_i^{2(2m-1)} \right] \sigma^m(a^2) a^{\text{inv}(s)} (-\bar{a})^{m(2m-1)-\text{inv}(s)} \\ &= \left[ \prod_{i=1}^m x_i^{2(2m-1)} \right] \sigma^m(a^2) (-a)^{-m(2m-1)} (-a^2)^{\text{inv}(s)}. \end{aligned}$$

---

<sup>2</sup>Kuperberg [4] considered only half-turn symmetric alternating-sign matrices of even order and denoted the partition function  $Z_{\text{HT}}(2m, \mathbf{x}, \mathbf{y})$  as  $Z_{\text{HT}}(m, \mathbf{x}, \mathbf{y})$ .

Thus, we have

$$C_{\text{HT}}(2m) = (-1)^m \sigma^m(a^2) a^{-m(2m-1)} \sum_{s \in S_{2m}^{\text{HT}}} (-a^2)^{\text{inv}(s)}.$$

Using relation (7), we come to equality (8).  $\square$

Singling out the terms of  $\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})$  which have the maximal total degree in all the variables  $x_i$  except the variable  $x_m$ , and do not contain all the variables  $y_i$  except the variable  $y_m$ , we write

$$\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^{m-1} x_i^{2(2m-1)} \right] S_{\text{HT}}(2m; x_m, y_m) + \dots$$

To find the polynomial  $S_{\text{HT}}(2m, x_m, y_m)$  we use the fact proved by Kuperberg [4] that the partition function  $Z_{\text{HT}}(2m, \mathbf{x}, \mathbf{y})$  is the product of two Laurent polynomials,

$$Z_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}) = Z(m; \mathbf{x}, \mathbf{y}) Z_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y}).$$

Hence, one can write

$$\tilde{Z}_{\text{HT}}(2m; \mathbf{x}, \mathbf{y}) = \tilde{Z}(m; \mathbf{x}, \mathbf{y}) \tilde{Z}_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y}). \quad (9)$$

**Lemma 10** *The function  $\tilde{Z}_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y})$  is a homogeneous polynomial in the variables  $x_i$  and  $y_i$  of total degree  $2m^2$ . For each fixed  $i = 1, \dots, m$  it is a polynomial in  $x_i^2$  of degree  $m$  and a polynomial in  $y_i^2$  of degree  $m$ . If  $y_m = ax_m$ , then*

$$\begin{aligned} \tilde{Z}_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y}) &= \sigma(a^2) x_m y_m \\ &\times \prod_{i=1}^{m-1} [(ay_i^2 - \bar{a}x_m^2)(ay_m^2 - \bar{a}x_i^2)] \tilde{Z}_{\text{HT}}^{(2)}(2(m-1); \mathbf{x} \setminus x_m, \mathbf{y} \setminus y_m). \end{aligned} \quad (10)$$

*Proof.* One can prove the lemma using relation (9) and Lemmas 8 and 3.  $\square$

**Lemma 11** *The polynomial  $S_{\text{HT}}(2m; x_m, y_m)$  has the form*

$$S_{\text{HT}}(2m; x_m, y_m) = S(m, x_m, y_m) S_{\text{HT}}^{(2)}(2m, x_m, y_m), \quad (11)$$

where  $S(m, x_m, y_m)$  is given by relation (5) and

$$S_{\text{HT}}^{(2)}(2m; x_m, y_m) = \left[ \prod_{i=1}^{m-1} \sigma(a^{2i-1}) \right] (\sigma(a^{2m-1}) x_m^{2m} + \sigma(a^{2m-3}) x_m^{2(m-1)} y_m^2). \quad (12)$$

*Proof.* Consider a state of square ice with half-turn symmetric boundary which gives a nontrivial contribution to the polynomial  $S_{\text{HT}}(2m; x_m, y_m)$ . It is clear that for such a state each line with the spectral parameter  $y_j$ ,  $j = 1, \dots, m-1$  may have only one vertex of first or second type, and the same is true for each line with the spectral parameter  $x_i$ ,  $i = 1, \dots, m-1$ . For the line with the spectral parameter  $x_m$  we have two possibilities. Either it has only one vertex of first or second type, or it has two vertices of first type and one of second type. In the latter case the line with the spectral parameter  $y_m$  also has two vertices of first type and one vertex of second type. Analysing all the possibilities one concludes that the polynomial  $S_{\text{HT}}(2m; x_m, y_m)$  has the form

$$S_{\text{HT}}(2m; x_m, y_m) = C_{\text{HT}}(2m) x_m^{2(2m-1)} + D_{\text{HT}}(2m) x_m^{2(2m-2)} y_m^2 + E_{\text{HT}}(2m) x_m^{2(2m-3)} y_m^4.$$

It follows from equality (9) that the polynomial  $S_{\text{HT}}(2m, x_m, y_m)$  can be represented in form (11) for some polynomial  $S_{\text{HT}}^{(2)}(2m; x_m, y_m)$ . Hence, having in mind (5), one can see that

$$S_{\text{HT}}^{(2)}(2m; x_m, y_m) = C_{\text{HT}}^{(2)}(2m)x_m^{2m} + D_{\text{HT}}^{(2)}(2m)x_m^{2(m-1)}y_m^2.$$

Using the evident equality

$$C_{\text{HT}}(2m) = C(m)C_{\text{HT}}^{(2)}(2m),$$

one obtains the relation

$$C_{\text{HT}}^{(2)}(2m) = \prod_{i=1}^m \sigma(a^{2i-1}). \quad (13)$$

Recursive relation (10) gives

$$C_{\text{HT}}^{(2)}(2m) + a^2 D_{\text{HT}}^{(2)}(2m) = \sigma(a^2)a^{-2m+3}C_{\text{HT}}^{(2)}(2(m-1)).$$

Taking into account equality (8), one comes to the relation

$$D_{\text{HT}}^{(2)}(2m) = - \left[ \prod_{i=1}^{m-1} \sigma(a^{2i-1}) \right] \sigma(a^{2m-3})$$

which implies equality (12).  $\square$

### 3.3 Square-ice model for matrices of odd order

Proceed now to the case of half-turn symmetric alternating-sign matrices of odd order. It is useful to have in mind that the central matrix element of such a matrix is either 1 or  $-1$ . The structure of the state pattern and the specification of the spectral parameters in this case can be understood from an example given in Figure 10. The cross means the change of the spectral parameter associated with the line. Here the direction of the arrow should be preserved. The following lemma is evident.

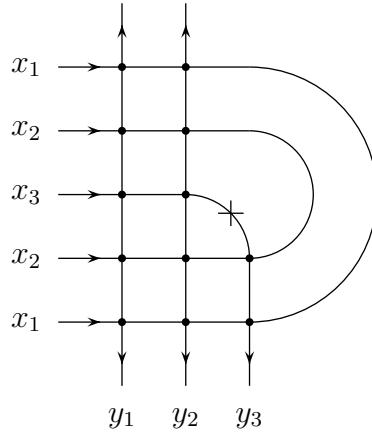


Figure 10: Square ice with a half-turn symmetric boundary of odd size

**Lemma 12** For a state of square ice with half-turn symmetric boundary of odd size corresponding to the permutation matrix associated with a permutation  $s \in S_{2m+1}^{\text{HT}}$ , the total number of the vertices of third type and of forth type is equal to  $\text{inv}(s)$  and the total number of the vertices of fifth type and of sixth type is equal to  $m(2m + 1) - \text{inv}(s)$ .

The partition function  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  is a Laurent polynomial in the variables  $x_i$  and  $y_i$ . It is convenient to introduce the modified partition function

$$\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^m x_i^{2m} y_i^{2m} \right] x_{m+1}^m y_{m+1}^m Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$$

which is a polynomial in the spectral parameters.

**Lemma 13** The partition function  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  is symmetric separately in the variables  $x_1, \dots, x_m$  and in the variables  $y_1, \dots, y_m$ . If  $y_1 = ax_1$ , then

$$\begin{aligned} Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}) &= \sigma^2(a^2) \sigma(a\bar{x}_1 y_{m+1}) \sigma(a\bar{x}_{m+1} y_1) \\ &\times \prod_{i=2}^m [\sigma^2(a\bar{x}_1 y_i) \sigma^2(a\bar{x}_i y_1)] Z_{\text{HT}}(2m - 1; \mathbf{x} \setminus x_1, \mathbf{y} \setminus y_1). \end{aligned} \quad (14)$$

*Proof.* To prove the symmetricity of the partition sum  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  in the variables  $x_1$  and  $x_2$  one multiplies it by  $\sigma(az)$ . The resulting expression may be considered as corresponding to the graph at the left-hand side of the equality given in Figure 11. If  $z = a\bar{x}_1 x_2$ , then using

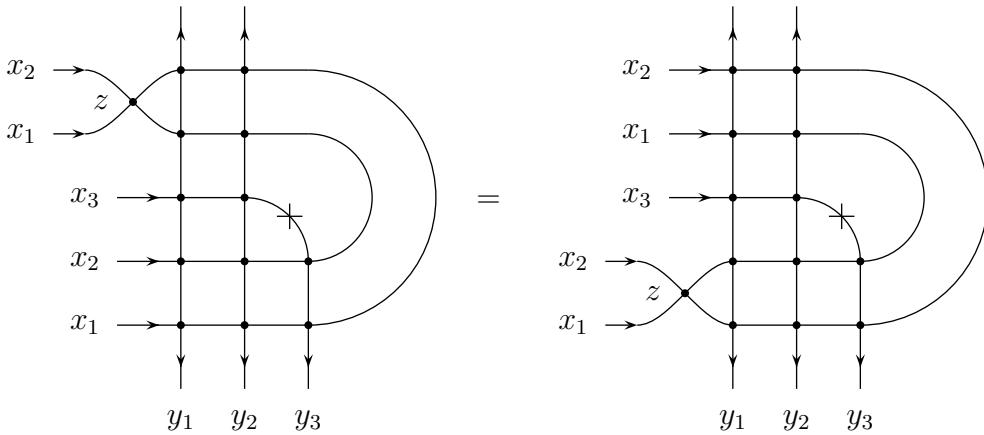


Figure 11: The proof of the symmetricity of  $Z_{\text{HT}}(3; \mathbf{x}, \mathbf{y})$  in  $x_1$  and  $x_2$

the Yang–Baxter equation, see Figure 4, one can move the crossing to the position given in the graph at the left-hand side of this equality. This proves the symmetricity in  $x_1$  and  $x_2$ . The other variables are treated in the same way.

To prove recursive relation (14) we note that if  $y_1 = ax_1$ , then only the states with a vertex of first type in the top-left corner give nonzero contribution to the partition function. Here the tetravalent vertices at the boundary of the graph become fixed, see Figure 12. They give all but last factors in the right-hand side of (14), and the remaining vertices give the last factor.  $\square$

**Lemma 14** The modified partition function  $\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  has the following properties.

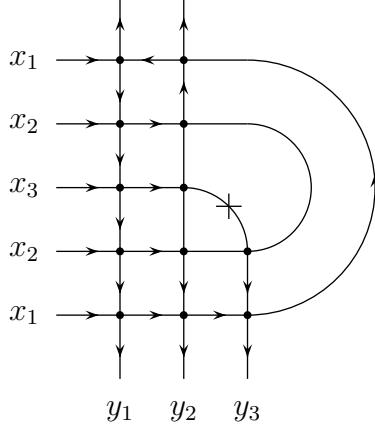


Figure 12: The proof of recursive relation (14)

- (a) The function  $\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  is symmetric separately in the variables  $x_1, \dots, x_m$  and in the variables  $y_1, \dots, y_m$ .
- (b) The function  $\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  is a homogeneous polynomial in the variables  $x_i$  and  $y_i$  of total degree  $2m(2m+1)$ . For each fixed  $i = 1, \dots, m$  it is a polynomial in  $x_i^2$  of degree  $2m$  and a polynomial in  $y_i^2$  of degree  $2m$ ; it is a polynomial of degree  $2m$  in  $x_{m+1}$  and a polynomial of degree  $2m$  in  $y_{m+1}$ .
- (c) If  $y_m = ax_m$ , then

$$\begin{aligned} \tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) &= \sigma^2(a^2)(ay_{m+1}^2 - \bar{a}x_m^2)(ay_m^2 - \bar{a}x_{m+1}^2)x_my_m \\ &\times \prod_{i=1}^{m-1} [(ay_i^2 - \bar{a}x_m^2)^2(ay_m^2 - \bar{a}x_i^2)^2] \tilde{Z}_{\text{HT}}(2m-1; \mathbf{x} \setminus x_m, \mathbf{y} \setminus y_m). \end{aligned} \quad (15)$$

*Proof.* The first two statements of the lemma are evident. Recursive relation (15) follows from the symmetricity of  $Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  and recursive relation (14).  $\square$

Singling out from  $\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  the term of maximal degree in the variables  $x_i$ , we write

$$\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^m x_i^{4m} \right] x_{m+1}^{2m} C_{\text{HT}}(2m+1) + \dots$$

**Lemma 15** The coefficient  $C_{\text{HT}}(2m+1)$  is given by the formula

$$C_{\text{HT}}(2m+1) = \prod_{i=2}^{2m+1} \sigma(a^i). \quad (16)$$

*Proof.* One can prove the lemma in the same way as Lemma 9 using Lemma 12 and equality (6).  $\square$

Singling out the terms of  $\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  which have the maximal total degree in all spectral parameters except  $x_{m+1}$  and  $y_{m+1}$ , we write

$$\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) = \left[ \prod_{i=1}^m x_i^{4m} \right] S_{\text{HT}}(2m+1; x_{m+1}, y_{m+1}) + \dots$$

The properties of the modified partition functions  $\tilde{Z}(n; \mathbf{x}, \mathbf{y})$  and  $\tilde{Z}_{\text{HT}}(2m, \mathbf{x}, \mathbf{y})$  described by Lemmas 3 and 8, respectively, determine them uniquely by Lagrange interpolation. It is not the case for the modified partition function  $\tilde{Z}_{\text{HT}}(2m + 1, \mathbf{x}, \mathbf{y})$ . Actually, what is missed here is the polynomial  $S_{\text{HT}}(2m + 1, x_{m+1}, y_{m+1})$ . To prove this fact we need the following lemma.

**Lemma 16** *The partition function  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  is invariant under the replacement  $x_i \rightarrow \bar{x}_i$ ,  $y_i \rightarrow \bar{y}_i$ . For the modified partition function one has*

$$\tilde{Z}_{\text{HT}}(2m + 1; \bar{\mathbf{x}}, \bar{\mathbf{y}}) = \left[ \prod_{i=1}^m x_i^{-4m} y_i^{-4m} \right] x_{m+1}^{-2m} y_{m+1}^{-2m} \tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}), \quad (17)$$

where  $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_{m+1})$  and  $\bar{\mathbf{y}} = (\bar{y}_1, \dots, \bar{y}_{m+1})$ .

*Proof.* Consider a state of square ice with a half-turn symmetric boundary. Reflect the corresponding graph through a horizontal line, and then rotate the half-line with the spectral parameter  $y_{m+1}$  by  $180^\circ$ . It is clear that the weight of the new state is obtained from the weight of the old one by the substitution  $x_i \rightarrow \bar{x}_i$ ,  $y_i \rightarrow \bar{y}_i$ . This fact implies the invariance of  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  under this substitution. Equality (17) follows immediately from the invariance of  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$ .  $\square$

**Lemma 17** *If two functions  $\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  and  $\tilde{Z}'_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  have the properties described in Lemma 14, and the corresponding polynomials  $S_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1})$  and  $S'_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1})$  coincide, then  $\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}) = \tilde{Z}'_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$ .*

*Proof.* Using statements (a) and (b) of Lemma 14, one can see that the difference of  $\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  and  $\tilde{Z}'_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  is zero when  $y_j = ax_i$ . Since  $\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  and  $\tilde{Z}'_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  are polynomial in  $y_i^2$ , this difference is also zero when  $y_j = -ax_i$ . Using (17), one can put recursive relation (15) into the form valid for  $y_1 = \bar{a}x_1$ . This form of the recursive relation (15) and statement (a) of Lemma 14 implies that the difference of  $\tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  and  $\tilde{Z}'_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$  is zero when  $y_j = \bar{a}x_i$  and  $y_j = \bar{a}x_i$ . Using these facts one easily obtains the equality

$$\begin{aligned} \tilde{Z}_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}) - \tilde{Z}'_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}) &= \prod_{i,j=1}^m [(y_i^2 - a^2 x_j)(y_i^2 - \bar{a}^2 x_j)] \\ &\times (S_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1}) - S'_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1})), \end{aligned}$$

which makes the statement of the lemma evident.  $\square$

### 3.4 Additional recursive relation

To find the polynomial  $S_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1})$  we use an additional recursive relation satisfied by the partition function  $Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y})$ .

**Lemma 18** *If  $y_{m+1} = ax_{m+1}$ , then*

$$Z_{\text{HT}}(2m + 1; \mathbf{x}, \mathbf{y}) = \prod_{i=1}^m [\sigma(a\bar{x}_i y_{m+1}) \sigma(a\bar{x}_{m+1} y_i)] Z_{\text{HT}}(2m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1}). \quad (18)$$

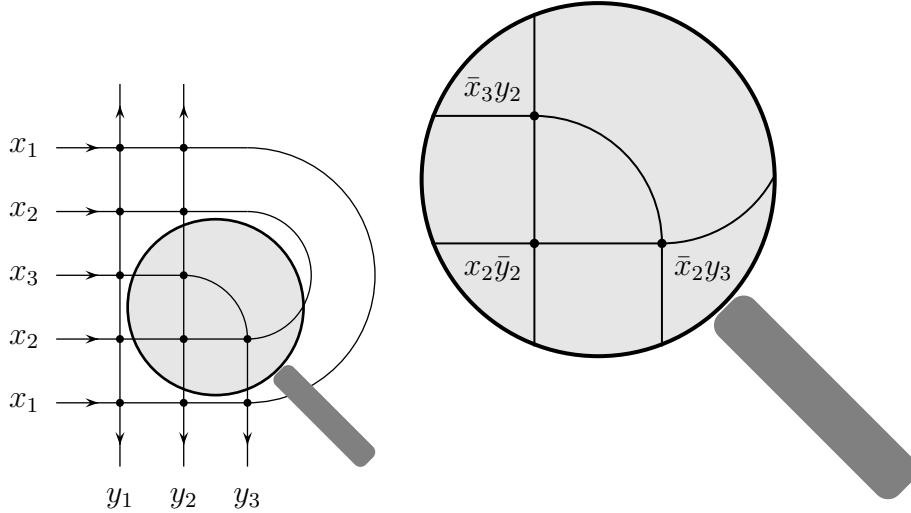


Figure 13: The central ‘triangle’ under a magnifying glass

*Proof.* Consider the spectral parameters of the vertices belonging to the central ‘triangle’ of the graph describing a state of square ice with a half-turn symmetric boundary, see example in Figure 13. It is clear that if  $y_{m+1} = ax_{m+1}$  then we can use the Yang–Baxter equation, given in Figure 4, and move the line with spectral parameters  $x_{m+1}$  and  $y_{m+1}$  to the boundary. An example of the process is given in Figure 14. Note that in the middle of the process the weights of moved vertices are not defined by the standard rule. Actually the roles of the parameters  $x_{m+1}$  and  $y_{m+1}$  interchange. In the final state the standard rule can be used if we interchange the spectral parameters  $x_{m+1}$  and  $y_{m+1}$ , as it is performed in Figure 14. The last graph in Figure 14 proves the statement of the lemma.  $\square$

The next lemma is a direct consequence of the previous one.

**Lemma 19** *If  $y_{m+1} = ax_{m+1}$ , then*

$$\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) = \prod_{i=1}^m [(ay_{m+1}^2 - \bar{a}x_i^2)(ay_i^2 - \bar{a}x_{m+1}^2)] \tilde{Z}_{\text{HT}}(2m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1}). \quad (19)$$

Now we are able to find the polynomial  $S_{\text{HT}}(2m+1, x_{m+1}, y_{m+1})$ .

**Lemma 20** *The polynomial  $S_{\text{HT}}(2m+1, x_{m+1}, y_{m+1})$  has the form*

$$S_{\text{HT}}(2m+1; x_{m+1}, y_{m+1}) = \left[ \prod_{i=2}^{2m} \sigma(a^i) \right] (\sigma(a^{2m+1})x_{m+1}^{2m} - \sigma(a^{2m})x_{m+1}^{2m-1}y_{m+1}). \quad (20)$$

*Proof.* Consider a state of square ice with a half-turn symmetric boundary which gives a non-trivial contribution to the polynomial  $S_{\text{HT}}(2m+1; x_m, y_m)$ . It is clear that for such a state each line with the spectral parameter  $y_j$ ,  $j = 1, \dots, m$  may have only one vertex of first or second type, and the same is true for each line with the spectral parameter  $x_i$ ,  $i = 1, \dots, m$ . For the line with the spectral parameters  $x_{m+1}$  and  $y_{m+1}$  we have two possibilities. Either it has no vertices of first or second type, or it has two vertices of first type. Actually in the former case one has

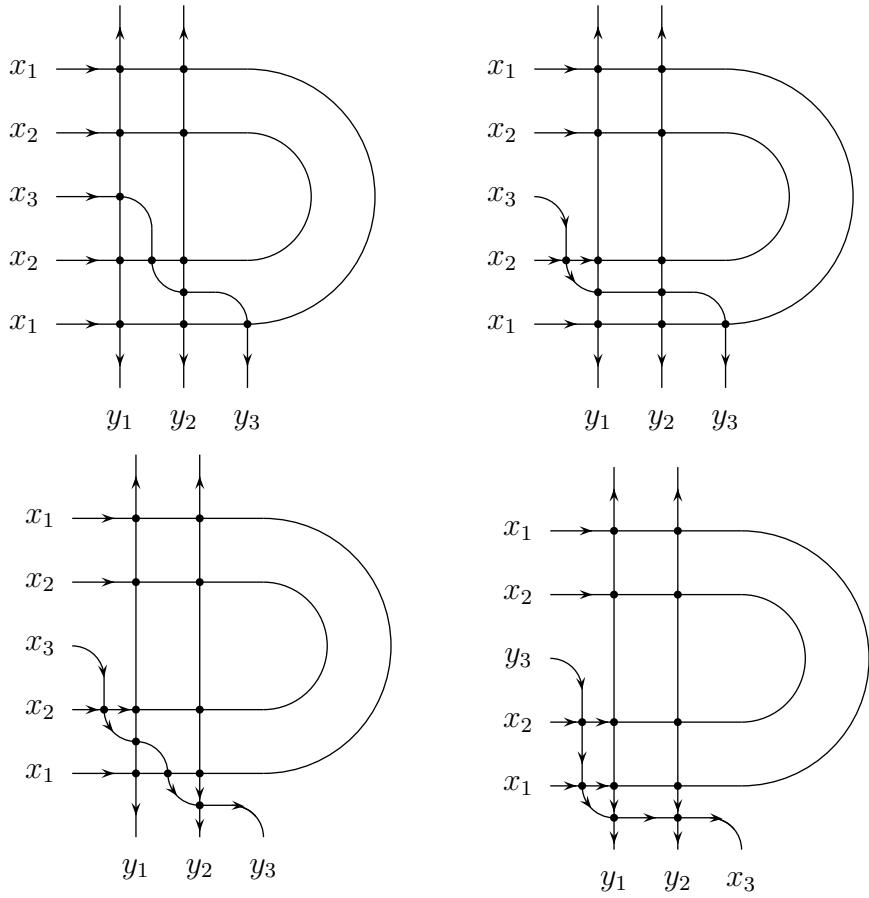


Figure 14: The proof of recursive relation (18)

a hidden vertex of first type at the turning-point and in the latter case it has a hidden vertex of second type there. One can get convinced that the polynomial  $S_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1})$  has the form

$$S_{\text{HT}}(2m + 1; x_{m+1}, y_{m+1}) = C_{\text{HT}}(2m + 1)x_{m+1}^{2m} + D_{\text{HT}}(2m + 1)x_{m+1}^{2m-1}y_{m+1}.$$

It follows from the recursive relation (19) that

$$C_{\text{HT}}(2m + 1) + aD_{\text{HT}}(2m + 1) = a^{-2m}C_{\text{HT}}(2m).$$

Using (16) one obtains the equality

$$D_{\text{HT}}(2m + 1) = - \left[ \prod_{i=2}^{2m} \sigma(a^i) \right] \sigma(a^{2m})$$

which immediately leads to (19).  $\square$

### 3.5 Main theorem

**Theorem 1** *The partition function for the square-ice model with half-turn symmetric boundary conditions can be represented as*

$$\begin{aligned} Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) &= \frac{ax_{m+1}y_{m+1}}{\sigma(a)(ax_{m+1} + y_{m+1})(ay_{m+1} + x_{m+1})} \\ &\times \left[ Z(m+1; \mathbf{x}, \mathbf{y})Z_{\text{HT}}^{(2)}(2m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1}) \right. \\ &\quad \left. + Z(m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1})Z_{\text{HT}}^{(2)}(2m+2; \mathbf{x}, \mathbf{y}) \right]. \end{aligned} \quad (21)$$

*Proof.* Consider first the modified partition function  $\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$ . The comparison of recursive relations (15) and (19) with recursive relations (2) and (10) suggests to use for it the following ansatz

$$\begin{aligned} \tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) &= A(x_{m+1}, y_{m+1})\tilde{Z}(m+1; \mathbf{x}, \mathbf{y})\tilde{Z}_{\text{HT}}^{(2)}(2m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1}) \\ &\quad + B(x_{m+1}, y_{m+1})\tilde{Z}(m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1})\tilde{Z}_{\text{HT}}^{(2)}(2m+2; \mathbf{x}, \mathbf{y}). \end{aligned}$$

This ansatz satisfies recursive relation (15) provided that the functional form of the coefficients  $A(x_{m+1}, y_{m+1})$  and  $B(x_{m+1}, y_{m+1})$  is the same for all  $m$ . Comparing the terms of the maximal total degree in all spectral parameters except  $x_{m+1}$  and  $y_{m+1}$  in both sides of the above equality, and using relations (20), (5), (13), (4) and (12), one comes to the condition

$$\begin{aligned} \frac{1}{\sigma(a)} [\sigma(a^{2m+1})x_{m+1}^2 - \sigma(a^{2m})x_{m+1}y_{m+1}] \\ = A(x_{m+1}, y_{m+1}) [\sigma(a^{2m+2})x_{m+1}^2 - \sigma(a^{2m})y_{m+1}^2] \\ + B(x_{m+1}, y_{m+1}) x_{m+1}^2 [\sigma(a^{2m+1})x_{m+1}^2 - \sigma(a^{2m-1})y_{m+1}^2] \end{aligned}$$

One can get convinced that this condition is satisfied for each  $m$  if and only if

$$\begin{aligned} A(x_m, y_m) &= \frac{ax_{m+1}y_{m+1}}{\sigma(a)(ax_{m+1} + y_{m+1})(ay_{m+1} + x_{m+1})}, \\ B(x_m, y_m) &= \frac{a}{\sigma(a)(ax_{m+1} + y_{m+1})(ay_{m+1} + x_{m+1})}. \end{aligned}$$

Taking into account Lemma 17, we conclude that we obtain a right expression for the modified partition function  $\tilde{Z}_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$ . Moving to functions without tildes we see that relation (21) is true.  $\square$

### 3.6 Two types of half-turn symmetric alternating-sign matrices

As we remarked above a half-turn symmetric alternating-sign matrix of odd order has either 1 or  $-1$  as its central entry. It appears that one can separate contribution of these two types of the matrices into the partition function. To this end note first that the weight of the vertices corresponding to 1 or  $-1$  entries of an alternating-sign matrix is an even function of the parameter  $a$ , while the weights of the vertices corresponding to 0 entries are odd functions of  $a$ . Let the number of  $-1$  entries in an  $n \times n$  alternating-sign matrix is equal to  $k$ , then the number of 1

entires in this matrix is equal to  $n + k$ . Hence, the number of its 0 entries is  $n(n - 1) - 2k$ . This number is always even and we have

$$Z(m; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a} = Z(m; \mathbf{x}, \mathbf{y}).$$

A similar consideration for the case of half-turn symmetric alternating-sign matrices of even order leads to the conclusion that

$$Z_{\text{HT}}(2m; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a} = (-1)^m Z_{\text{HT}}(m; \mathbf{x}, \mathbf{y}). \quad (22)$$

This equality, in particular, gives

$$Z_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a} = (-1)^m Z_{\text{HT}}^{(2)}(2m, \mathbf{x}, \mathbf{y}). \quad (23)$$

Consider now a half-turn symmetric alternating-sign matrix of odd order  $(2m+1) \times (2m+1)$ . Let the central entry of the matrix is 1. In this case the number of  $-1$  entries is even, say  $2l$ . Then the number of 0 entries is  $(2m+1)^2 - (2m+1) - 4l = 2m(2m+1) - 4l$ . Only a half of these vertices corresponds to vertices of the corresponding graph, describing the state of square-ice with a half-turn symmetric boundary. Therefore, the weight of the state under consideration acquires the factor  $(-1)^m$  under the replacement  $a \rightarrow -a$ . In a similar way we can see that the weight of the state corresponding to a half-turn symmetric alternating-sign matrix of odd order whose central entry is  $-1$  acquires the factor  $(-1)^{m+1}$  under this replacement. Thus, representing  $Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  in the form

$$Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) = Z_{\text{HT}}^{(+)}(2m+1; \mathbf{x}, \mathbf{y}) + Z_{\text{HT}}^{(-)}(2m+1; \mathbf{x}, \mathbf{y}),$$

where

$$Z_{\text{HT}}^{(+)}(2m+1; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a} = (-1)^m Z_{\text{HT}}^{(+)}(2m+1; \mathbf{x}, \mathbf{y}),$$

$$Z_{\text{HT}}^{(-)}(2m+1; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a} = (-1)^{m+1} Z_{\text{HT}}^{(-)}(2m+1; \mathbf{x}, \mathbf{y}),$$

we separate the contribution of two types of the matrices. It is clear that

$$\begin{aligned} Z_{\text{HT}}^{(+)}(2m+1; \mathbf{x}, \mathbf{y}) &= \frac{1}{2} [Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) + (-1)^m Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a}], \\ Z_{\text{HT}}^{(-)}(2m+1; \mathbf{x}, \mathbf{y}) &= \frac{1}{2} [Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y}) - (-1)^m Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})|_{a \rightarrow -a}]. \end{aligned}$$

Using relation (21) and taking into account equalities (22) and (23), one obtains the following theorem.

**Theorem 2** *The contributions to the partition function  $Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  of the states corresponding to the alternating-sign matrices having 1 or  $-1$  as the central entry are*

$$\begin{aligned} Z_{\text{HT}}^{(+)}(2m+1; \mathbf{x}, \mathbf{y}) &= \frac{1}{\sigma(a)\sigma(ax_{m+1}\bar{y}_{m+1})\sigma(a\bar{x}_{m+1}y_{m+1})} \\ &\times \left[ (a + \bar{a})Z(m+1; \mathbf{x}, \mathbf{y})Z_{\text{HT}}^{(2)}(2m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1}) \right. \\ &\quad \left. - (x_{m+1}\bar{y}_{m+1} + \bar{x}_{m+1}y_{m+1})Z(m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1})Z_{\text{HT}}^{(2)}(2m+2; \mathbf{x}, \mathbf{y}) \right], \quad (24) \end{aligned}$$

$$\begin{aligned} Z_{\text{HT}}^{(-)}(2m+1; \mathbf{x}, \mathbf{y}) &= \frac{1}{\sigma(a)\sigma(ax_{m+1}\bar{y}_{m+1})\sigma(a\bar{x}_{m+1}y_{m+1})} \\ &\times \left[ -(x_{m+1}\bar{y}_{m+1} + \bar{x}_{m+1}y_{m+1})Z(m+1; \mathbf{x}, \mathbf{y})Z_{\text{HT}}^{(2)}(2m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1}) \right. \\ &\quad \left. + (a + \bar{a})Z(m; \mathbf{x} \setminus x_{m+1}, \mathbf{y} \setminus y_{m+1})Z_{\text{HT}}^{(2)}(2m+1; \mathbf{x}, \mathbf{y}) \right] \quad (25) \end{aligned}$$

respectively.

## 4 Special determinant representations

It appears that in the case  $a = e^{i\pi/3}$  partition functions of many square-ice models possess additional symmetry properties and new representations. It is interesting to consider from this point of view the square-ice model with half-turn symmetric boundary condition. Assume that  $a = e^{i\pi/3}$  and discuss first additional properties of the partition functions  $Z(n; \mathbf{x}, \mathbf{y})$  and  $Z_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y})$  arising in this case.

Consider the partition function  $Z(n; \mathbf{x}, \mathbf{y})$  as a function of the  $2n$ -dimensional vector  $\mathbf{u} = (u_1, \dots, u_{2n})$ , where  $u_{2i-1} = x_i$  and  $u_{2i} = y_i$ . From recursive relation (3) one obtains that if  $u_{2n} = au_{2n-1}$ , then

$$Z(n; \mathbf{u}) = \sigma(a) \prod_{\mu=1}^{2n-2} \sigma(a u_\mu \bar{u}_{2n-1}) Z(n-1; \mathbf{u} \setminus u_{2n-1} \setminus u_{2n}). \quad (26)$$

It can be shown that for any  $\mu = 1, \dots, 2n$  the partition function  $Z(n; \mathbf{u})$  satisfies also the relation

$$\begin{aligned} & Z(n; (u_1, \dots, u_\mu, \dots, u_{2n})) \prod_{\nu \neq \mu} \sigma(u_\nu \bar{u}_\mu) \\ & + Z(n; (u_1, \dots, a^2 u_\mu, \dots, u_{2n})) \prod_{\nu \neq \mu} \sigma(u_\nu \bar{a}^2 \bar{u}_\mu) \\ & + Z(n; (u_1, \dots, \bar{a}^2 u_\mu, \dots, u_{2n})) \prod_{\nu \neq \mu} \sigma(u_\nu a^2 \bar{u}_\mu) = 0. \end{aligned} \quad (27)$$

These relations allow one to obtain the following determinant representation for the partition function  $Z(n; \mathbf{u})$ :

$$Z(n; \mathbf{u}) = (-1)^{n(n-1)/2} \frac{\sigma^n(a)}{\prod_{\mu < \nu} \sigma(u_\mu \bar{u}_\nu)} \det P(n, \mathbf{u}), \quad (28)$$

where

$$P(n; \mathbf{u}) = \begin{pmatrix} u_1^{3n-2} & u_2^{3n-2} & u_3^{3n-2} & \dots & u_{2n}^{3n-2} \\ u_1^{3n-4} & u_2^{3n-4} & u_3^{3n-4} & \dots & u_{2n}^{3n-4} \\ u_1^{3n-8} & u_2^{3n-8} & u_3^{3n-8} & \dots & u_{2n}^{3n-8} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1^{-3n+2} & u_2^{-3n+2} & u_3^{-3n+2} & \dots & u_{2n}^{-3n+2} \end{pmatrix},$$

see [9, 10]. It follows from the above representation that the partition function  $Z(n; \mathbf{u})$  is symmetric in the coordinates of the vector  $\mathbf{u}$ .

For the function  $Z_{\text{HT}}^{(2)}(2m; \mathbf{x}, \mathbf{y})$  considered as a function of the  $2m$ -dimensional vector  $\mathbf{u} = (u_1, \dots, u_{2m})$ , at  $a = e^{i\pi/3}$  one obtains actually the same relations

$$Z_{\text{HT}}^{(2)}(2m; \mathbf{u}) = \sigma(a) \prod_{\mu=1}^{2m-2} \sigma(a u_\mu \bar{u}_{2m-1}) Z_{\text{HT}}^{(2)}(2m-2; \mathbf{u} \setminus u_{2m-1} \setminus u_{2m}) \quad (29)$$

and

$$\begin{aligned}
& Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, u_\mu, \dots, u_{2m})) \prod_{\nu \neq \mu} \sigma(u_\nu \bar{u}_\mu) \\
& + Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, a^2 u_\mu, \dots, u_{2m})) \prod_{\nu \neq \mu} \sigma(u_\nu \bar{a}^2 \bar{u}_\mu) \\
& + Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, \bar{a}^2 u_\mu, \dots, u_{2m})) \prod_{\nu \neq \mu} \sigma(u_\nu a^2 \bar{u}_\mu) = 0, \quad (30)
\end{aligned}$$

as for the partiton function  $Z(n, \mathbf{u})$ . They give the determinant representation [11]

$$Z_{\text{HT}}^{(2)}(2m; \mathbf{u}) = (-1)^{m(m-1)/2} \frac{\sigma^m(a)}{\prod_{\mu < \nu} \sigma(u_\mu \bar{u}_\nu)} \det Q(m, \mathbf{u}), \quad (31)$$

where

$$Q(m; \mathbf{u}) = \begin{pmatrix} u_1^{3m-1} & u_2^{3m-1} & u_3^{3m-1} & \dots & u_{2m}^{3m-1} \\ u_1^{3m-5} & u_2^{3m-5} & u_3^{3m-5} & \dots & u_{2m}^{3m-5} \\ u_1^{3m-7} & u_2^{3m-7} & u_3^{3m-7} & \dots & u_{2m}^{3m-7} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1^{-3m+1} & u_2^{-3m+1} & u_3^{-3m+1} & \dots & u_{2m}^{-3m+1} \end{pmatrix}.$$

We again have symmetricity in the coordinates of the vector  $\mathbf{u}$ . Therefore, the partition function  $Z_{\text{HT}}(2m, \mathbf{x}, \mathbf{y})$  at  $a = e^{i\pi/3}$  is symmetric in the union of the coordinates of the vectors  $\mathbf{x}$  and  $\mathbf{y}$ .

From the other hand, the partition function  $Z_{\text{HT}}(2m+1; \mathbf{x}, \mathbf{y})$  at  $a = e^{i\pi/3}$  is not symmetric in the union of the coordinates of the vectors  $\mathbf{x}$  and  $\mathbf{y}$ . Assume again that  $a = e^{i\pi/3}$  and consider the function

$$Z'_{\text{HT}}(2m+1; \mathbf{u}) = Z_{\text{HT}}(2m+1; (u_1, u_3, \dots, u_{2m-1}, u_{2m+1}), (u_2, u_4, \dots, u_{2m}, u_{2m+1})),$$

where  $\mathbf{u} = (u_1, \dots, u_{2m+1})$ . As it follows from the next theorem the function  $Z'_{\text{HT}}(2m+1; \mathbf{u})$  is symmetric in the coordinates of the vector  $\mathbf{u}$ .

**Theorem 3** *The function  $Z'_{\text{HT}}(2m+1; \mathbf{u})$  has the following determinant representation*

$$Z'_{\text{HT}}(2m+1, \mathbf{u}) = \frac{\sigma^{2m}(a)}{\prod_{\mu < \nu} \sigma^2(u_\mu \bar{u}_\nu)} \det P'(m+1; \mathbf{u}) \det P'(m+1; \bar{\mathbf{u}}), \quad (32)$$

where  $P'(m, \mathbf{u})$  is the  $(2m-1) \times (2m-1)$  matrix which is obtained from  $P(m, \mathbf{u})$  when one removes the last column and the last row.

*Proof.* Equality (21) implies

$$\begin{aligned}
& Z'_{\text{HT}}(2m+1; \mathbf{u}) = -\frac{1}{\sigma^3(a)} \\
& \times \left[ Z(m+1; (u_1, \dots, u_{2m}, u_{2m+1}, u_{2m+1})) Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, u_{2m})) \right. \\
& \left. + Z_{\text{HT}}^{(2)}(2m+2; (u_1, \dots, u_{2m}, u_{2m+1}, u_{2m+1})) Z(m; (u_1, \dots, u_{2m})) \right]. \quad (33)
\end{aligned}$$

Let us multiply equation (27) for  $n = m$  and  $\mu = 2m$  by  $Z_{\text{HT}}(m; (u_1, \dots, \bar{a}^2 u_{2m}))$ , equation (30) for  $\mu = 2m$  by  $Z(m; (u_1, \dots, \bar{a}^2 u_{2m}))$  and find the difference of the obtained expressions. The result can be written as

$$\frac{W(m; (u_1, \dots, u_{2m-1}, u_{2m}))}{\prod_{\nu=1}^{2m-1} \sigma(u_\nu \bar{u}_{2m})} = \frac{W(m; (u_1, \dots, u_{2m-1}, a^2 u_{2m}))}{\prod_{\nu=1}^{2m-1} \sigma(u_\nu \bar{a}^2 \bar{u}_{2m})}, \quad (34)$$

where we introduced the ‘Wronskian’

$$W(m; \mathbf{u}) = Z(m; (u_1, \dots, u_{2m-1}, \bar{a}^2 u_{2m})) Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, u_{2m-1}, a^2 u_{2m})) \\ - Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, u_{2m-1}, \bar{a}^2 u_{2m})) Z(m; (u_1, \dots, u_{2m-1}, a^2 u_{2m})).$$

The function  $W(m; \mathbf{u})$  is a centered Laurent polynomial in  $u_{2m}$  of width  $2m - 1$ . The product  $\prod_{\mu=1}^{2m-1} \sigma(u_\mu \bar{u}_{2m})$  is also a centered Laurent polynomial of the same width. Multiplying the numerators and denominators of the fractions in both sides of equality (34) by  $u_{2m}^{2m-1}$ , we see that these fractions are rational functions of  $u_{2m}^2$ . The positions of possible poles in  $u_{2m}^2$  of the left-hand and right-hand sides of (34) are different. This means that the equality can be true only if the rational functions under consideration are constant in  $u_{2m}^2$ . Thus, one has

$$W(m; \mathbf{u}) = w(m; \mathbf{u} \setminus u_{2m}) \prod_{\mu=1}^{2m-1} \sigma(u_\mu \bar{u}_{2m}). \quad (35)$$

Using the equalities

$$Z(m; (u_1, \dots, -u_\mu, \dots, u_{2m})) = (-1)^{m-1} Z(m; (u_1, \dots, u_\mu, \dots, u_{2m})), \\ Z_{\text{HT}}^{(2)}(m; (u_1, \dots, -u_\mu, \dots, u_{2m})) = (-1)^m Z_{\text{HT}}^{(2)}(m; (u_1, \dots, u_\mu, \dots, u_{2m})),$$

and recursive relations (26) and (29), one can get convinced that

$$W(m; (u_1, \dots, u_{2m-2}, u_{2m-1}, a^2 u_{2m-1})) = (-1)^m \sigma(a) \prod_{\mu=1}^{2m-2} \sigma(a u_\mu \bar{u}_{2m-1}) \\ \times \left[ Z(m; (u_1, \dots, u_{2m-2}, u_{2m-1}, u_{2m-1})) Z_{\text{HT}}^{(2)}(2m-2; (u_1, \dots, u_{2m-2})) \right. \\ \left. + Z_{\text{HT}}^{(2)}(2m; (u_1, \dots, u_{2m-2}, u_{2m-1}, u_{2m-1})) Z(m-1; (u_1, \dots, u_{2m-2})) \right].$$

From the other hand, it follows from (35) that

$$W(m; (u_1, \dots, u_{2m-1}, a^2 u_{2m-1})) = -\sigma(a) \prod_{\mu=1}^{2m-2} \sigma(a u_\mu \bar{u}_{2m-1}) w(m, (u_1, \dots, u_{2m-1})). \quad (36)$$

The above two equalities and (33) give

$$Z_{\text{HT}}(2m+1; \mathbf{u}) = \frac{(-1)^{m+1}}{\sigma^3(a)} w(m+1; \mathbf{u}). \quad (37)$$

From the determinant representations (28) and (31) one obtains that as  $u_{2m} \rightarrow 0$

$$Z(m, \mathbf{u}) \sim \frac{1}{u_{2m}^{m-1}} \frac{(-1)^{m(m-1)/2} \sigma^m(a)}{\prod_{\mu=1}^{2m-1} u_\mu \prod_{\substack{\mu, \nu=1 \\ \mu < \nu}}^{2m-1} \sigma(u_\mu \bar{u}_\nu)} \det P'(m; \mathbf{u} \setminus u_{2m}) + \dots,$$

$$Z_{\text{HT}}^{(2)}(2m, \mathbf{u}) \sim \frac{1}{u_{2m}^m} \frac{(-1)^{m(m-1)/2} \sigma^m(a)}{\prod_{\mu=1}^{2m-1} u_\mu \prod_{\substack{\mu, \nu=1 \\ \mu < \nu}}^{2m-1} \sigma(u_\mu \bar{u}_\nu)} \det Q'(m; \mathbf{u} \setminus u_{2m}) + \dots,$$

where  $Q'(m, \mathbf{u})$  is the  $(2m-1) \times (2m-1)$  matrix which is obtained from  $Q(m, \mathbf{u})$  when one removes the last column and the last row. These relations show that

$$W(m, \mathbf{u}) \sim -\frac{1}{u_{2m}^{2m-1}} \frac{\sigma^{2m+1}(a)}{\prod_{\mu=1}^{2m-1} u_\mu^2 \prod_{\substack{\mu, \nu=1 \\ \mu < \nu}} \sigma^2(u_\mu \bar{u}_\nu)} \det P'(m; \mathbf{u} \setminus u_{2m}) \det Q'(m; \mathbf{u} \setminus u_{2m}) + \dots$$

as  $u_{2m} \rightarrow 0$ . Using the equality

$$\det P'(m, \bar{\mathbf{u}}) = (-1)^m \prod_{\mu=1}^{2m-1} u_\mu^{-3} \det Q'(m, \mathbf{u})$$

and relation (36), we obtain

$$w(m, \mathbf{u}) = (-1)^m \frac{\sigma^{2m+1}(a)}{\prod_{\mu < \nu} \sigma^2(u_\mu \bar{u}_\nu)} \det P'(m; \mathbf{u}) \det P'(m; \bar{\mathbf{u}}).$$

The statement of the lemma follows now from (37).  $\square$

Using quite different technique, Okada also obtained the determinant representations (28) and (31) [12]. He expressed the results in terms of characters of classical groups and conjectured, in particular, that the number the half-turn symmetric alternating-sign matrices of odd order is connected with the dimension of some specific representation of  $\mathrm{GL}(2n+1) \times \mathrm{GL}(2n+1)$ . Possible generalizations of this conjecture have been discussed by Kuperberg [13]. We hope that our determinant representation (32) sheds a new light on this question.

## 5 Enumeration results

### 5.1 Refined $x$ -enumerations

Denote by  $A(n; x)$  the total weight of the  $n \times n$  alternating-sign matrices, where the weight of an individual alternating-sign matrix is  $x^k$  if it has  $k$  matrix elements equal to  $-1$ . The quantities  $A(n; x)$  are called  $x$ -enumerations of the alternating-sign matrices.<sup>3</sup> Considering the partition function  $Z(n; \mathbf{x}, \mathbf{y})$  at  $\mathbf{x} = \mathbf{y} = \mathbf{1}$ , where  $\mathbf{1} = (1, \dots, 1)$ , one obtains information on  $x$ -enumerations. Namely, one has the equality

$$A(n; x) = \frac{1}{\sigma^{n^2-n}(a)\sigma^n(a^2)} Z(n; \mathbf{1}, \mathbf{1}),$$

where

$$x = \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^2 = (a + \bar{a})^2.$$

In particular, if  $a = e^{i\pi/3}$ , then  $x = 1$ , and the above equality gives the total number of the  $n \times n$  alternating-sign matrices  $A(n) = A(n, 1)$ . In paper [3], Kuperberg used the Izergin–Korepin determinant representation for the partition function  $Z(n; \mathbf{x}, \mathbf{y})$  [14, 15] and proved the formula for  $A(n)$  conjectured by Mills, Robbins and Rumsey [1, 2] and first proved by Zeilberger [16].

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<sup>3</sup>Do not mix the vector  $\mathbf{x}$  and the parameter of the enumeration  $x$ .

One also defines refined  $x$ -enumerations of the alternating-sign matrices. Recall that an alternating-sign matrix has only one entry 1 in the first column, all other entries are zero. Denote by  $A(n, r; x)$  the total weight of the  $n \times n$  alternating-sign matrices having 1 at the  $r$ th position of the first column. Here the weight of an individual alternating-sign matrix is  $x^k$  if it has  $k$  matrix elements equal to  $-1$ . It is readily seen that we have the equality

$$\mathcal{A}(n; t, x) \equiv \sum_{r=1}^n A(n, r; x) t^{r-1} = \frac{Z(n; \mathbf{1}, (v, 1, \dots, 1))}{\sigma^{n^2-2n+1}(a) \sigma^n(a^2) \sigma^{n-1}(av)}, \quad (38)$$

where

$$x = \left[ \frac{\sigma(a^2)}{\sigma(a)} \right]^2, \quad t = \frac{\sigma(a\bar{v})}{\sigma(av)}.$$

If  $a = e^{i\pi/3}$  one obtains formulas for refined enumerations of the alternating-sign matrices  $A(n, r) = A(n, r; 1)$ . In paper [17], Zeilberger used the Izergin–Korepin determinant representation for  $Z(n; \mathbf{x}, \mathbf{y})$  to prove the refined alternating-sign matrix conjecture for  $A(n, r)$  by Mills, Robbins, and Rumsey [1, 2].

For the case of half-turn symmetric alternating-sign matrices one has

$$\mathcal{A}_{\text{HT}}(2m; t, x) \equiv \sum_{r=1}^{2m} A_{\text{HT}}(2m, r; x) t^{r-1} = \frac{Z_{\text{HT}}(2m; \mathbf{1}, (v, 1, \dots, 1))}{\sigma^{2m^2-3m+1}(a) \sigma^m(a^2) \sigma^{2m-1}(av)}.$$

Note that here the weight of an individual alternating-sign matrix is  $x^{k/2}$  if it has  $k$  entries equal to  $-1$ . Since in the case under consideration the number of  $-1$  entries is always even, we weigh in accordance with the number of symmetry orbits of  $-1$  entries. It is convenient to introduce the notation

$$\mathcal{A}_{\text{HT}}^{(2)}(2m; t, x) \equiv \frac{\mathcal{A}_{\text{HT}}(2m; t, x)}{\mathcal{A}(m; t, x)} = \frac{Z_{\text{HT}}^{(2)}(2m; \mathbf{1}, (v, 1, \dots, 1))}{\sigma^{m^2-m}(a) \sigma^m(av)}. \quad (39)$$

Further, for the case of the half-turn symmetric alternating-sign matrices of odd order we obtain

$$\mathcal{A}_{\text{HT}}(2m+1; t, x) \equiv \sum_{r=1}^{2m+1} A_{\text{HT}}(2m+1, r; x) t^{r-1} = \frac{Z_{\text{HT}}(2m+1; \mathbf{1}, (v, 1, \dots, 1))}{\sigma^{2m^2-m}(a) \sigma^m(a^2) \sigma^{2m}(av)}. \quad (40)$$

Here again the weight of an individual alternating-sign matrix is  $x^{k/2}$  if it has  $k$  entries equal to  $-1$ . Note that the number of  $-1$  entries is even or odd, if the central entry of the matrix is 1 or  $-1$  respectively. Robbins used the weighing in accordance with the number of symmetry orbits of  $-1$  entries in this case as well [5]. Our definition seems more convenient. The connection with the  $x$ -enumeration used by Robbins is given below.

Having in mind representation (21), from (40), (38) and (39) we obtain

$$\mathcal{A}_{\text{HT}}(2m+1; t, x) = \frac{\sqrt{x} \mathcal{A}(m+1; t, x) \mathcal{A}_{\text{HT}}^{(2)}(2m; t, x) + \mathcal{A}(m; t, x) \mathcal{A}_{\text{HT}}^{(2)}(2m+2; t, x)}{\sqrt{x} + 2}. \quad (41)$$

While  $\mathcal{A}(m; t, x)$  and  $\mathcal{A}_{\text{HT}}^{(2)}(m; t, x)$  are polynomials in the variables  $x$  and  $t$ ,  $\mathcal{A}_{\text{HT}}(2m+1; t, x)$  has also half-integer powers of the variable  $x$ . One can separate it into two parts

$$\mathcal{A}_{\text{HT}}(2m+1; t, x) = \mathcal{A}_{\text{HT}}^{(+)}(2m+1; t, x) + \sqrt{x} \mathcal{A}_{\text{HT}}^{(-)}(2m+1; t, x),$$

where  $\mathcal{A}_{\text{HT}}^{(+)}(2m+1, t, x)$  and  $\mathcal{A}_{\text{HT}}^{(-)}(2m+1; t, x)$  are polynomials in the variable  $x$ . These two parts give the refined  $x$ -enumerations of the half-turn symmetric alternating-sign matrices of odd order with 1 and  $-1$  in the centre of a matrix respectively. Relation (41) gives<sup>4</sup>

$$\begin{aligned} \mathcal{A}_{\text{HT}}^{(+)}(2m+1; t, x) \\ = \frac{-x\mathcal{A}(m+1; t, x)\mathcal{A}_{\text{HT}}^{(2)}(2m; t, x) + 2\mathcal{A}(m; t, x)\mathcal{A}_{\text{HT}}^{(2)}(2m+2; t, x)}{4-x}, \end{aligned} \quad (42)$$

$$\begin{aligned} \mathcal{A}_{\text{HT}}^{(-)}(2m+1; t, x) \\ = \frac{2\mathcal{A}(m+1; t, x)\mathcal{A}_{\text{HT}}^{(2)}(2m; t, x) - \mathcal{A}(m; t, x)\mathcal{A}_{\text{HT}}^{(2)}(2m+2; t, x)}{4-x}. \end{aligned} \quad (43)$$

Robbins used the  $x$ -enumeration of the half-turn symmetric alternating-sign matrices related to the number of symmetry orbits of  $-1$  entries [5]. It is clear that such  $x$ -enumeration has the form

$$\begin{aligned} \mathcal{A}_{\text{HT}}^{\text{R}}(2m+1; t, x) &= \mathcal{A}_{\text{HT}}^{(+)}(2m+1; t, x) + x\mathcal{A}_{\text{HT}}^{(-)}(2m+1; t, x) \\ &= \frac{x\mathcal{A}(m+1; t, x)\mathcal{A}_{\text{HT}}^{(2)}(2m; t, x) + (2-x)\mathcal{A}(m; t, x)\mathcal{A}_{\text{HT}}^{(2)}(2m+2; t, x)}{4-x}. \end{aligned}$$

We see that the refined  $x$ -enumerations  $\mathcal{A}_{\text{HT}}^{(+)}(2m+1; t, x)$  and  $\mathcal{A}_{\text{HT}}^{(-)}(2m+1; t, x)$  are determined by the polynomials  $\mathcal{A}(m; t, x)$  and  $\mathcal{A}_{\text{HT}}^{(2)}(2m; t, x)$  (or  $\mathcal{A}_{\text{HT}}(2m; t, x)$ ). These polynomials for general  $x$  are not known yet. However, for  $x = 1$  we know the explicit form of these polynomials. This allows us to obtain the expressions for the refined 1-enumerations of the haft-turn symmetric alternating-sign matrices of odd order. We start with the usual enumerations (1-enumerations).

## 5.2 1-enumerations

Recall that the total number of the alternating-sign matrices is given by the formula

$$A(m) = \prod_{i=0}^{m-1} \frac{(3i+1)!}{(m+i)!},$$

and for the total number of half-turn symmetric alternating-sign matrices one has

$$A_{\text{HT}}(2m) = \prod_{i=0}^{m-1} \frac{(3i)!(3i+2)!}{[(m+i)!]^2}.$$

It follows from these relations that

$$\frac{A(m+1)}{A(m)} = \frac{m!(3m+1)!}{(2m)!(2m+1)!}, \quad (44)$$

$$\frac{A_{\text{HT}}(2m+2)}{A_{\text{HT}}(2m)} = \frac{[m!]^2 (3m)!(3m+2)!}{[(2m)!(2m+1)!]^2}. \quad (45)$$

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<sup>4</sup>Certainly, one can directly use relations (24) and (25).

Putting  $x = 1$  and  $t = 1$  and having in mind that  $\mathcal{A}(n; 1, 1) = A(n)$  and  $\mathcal{A}_{\text{HT}}^{(2)}(2m; 1, 1) = A_{\text{HT}}(2m)/A(m)$ , we obtain from (42) and (43) that

$$\begin{aligned}\frac{\mathcal{A}_{\text{HT}}^{(+)}(2m+1)}{A_{\text{HT}}(2m)} &= -\frac{1}{3} \frac{A(m+1)}{A(m)} + \frac{2}{3} \frac{A_{\text{HT}}(2m+2)}{A_{\text{HT}}(2m)} \frac{A(m)}{A(m+1)}, \\ \frac{\mathcal{A}_{\text{HT}}^{(-)}(2m+1)}{A_{\text{HT}}(2m)} &= \frac{2}{3} \frac{A(m+1)}{A(m)} - \frac{1}{3} \frac{A_{\text{HT}}(2m+2)}{A_{\text{HT}}(2m)} \frac{A(m)}{A(m+1)}.\end{aligned}$$

Using relations (44) and (45), one can express  $\mathcal{A}_{\text{HT}}^{(+)}(2m+1)$  and  $\mathcal{A}_{\text{HT}}^{(-)}(2m+1)$  through  $A_{\text{HT}}(2m)$ . It is convenient to write the answer in the following form

$$\begin{aligned}A_{\text{HT}}(2m+1) &= \frac{(m)!(3m)!}{[(2m)!]^2} A_{\text{HT}}(2m), \\ \mathcal{A}_{\text{HT}}^{(+)}(2m+1) &= \frac{m+1}{2m+1} A_{\text{HT}}(2m+1), \quad \mathcal{A}_{\text{HT}}^{(-)}(2m+1) = \frac{m}{2m+1} A_{\text{HT}}(2m+1).\end{aligned}$$

The first equality was conjectured by Robbins [5] as well, but, as far as we know, has not been proved yet.

The simplicity of the relation

$$\frac{\mathcal{A}_{\text{HT}}^{(+)}(2m+1)}{\mathcal{A}_{\text{HT}}^{(-)}(2m+1)} = \frac{m+1}{m}$$

is rather unexpected.

### 5.3 Refined 1-enumerations

The polynomial  $\mathcal{A}(m; t) \equiv \mathcal{A}(m; t, 1)$  is determined by the celebrated refined enumeration of the alternating-sign matrices conjectured by Mills, Robbins, and Rumsey [1, 2] and proved by Zeilberger [17]. It has the form

$$\frac{\mathcal{A}(m, t)}{A(m)} = \frac{(2m-1)!}{(m-1)!(3m-2)!} \sum_{r=1}^m \frac{(m+r-2)!(2m-r-1)!}{(r-1)!(m-r)!} t^{r-1}.$$

The polynomial  $\mathcal{A}_{\text{HT}}^{(2)}(m; t) \equiv \mathcal{A}_{\text{HT}}^{(2)}(m; t, 1)$  was found in paper [11]. It is given by the formula

$$\begin{aligned}\frac{\mathcal{A}_{\text{HT}}^{(2)}(2m; t)}{A_{\text{HT}}^{(2)}(2m)} &= \frac{(3m-2)(2m-1)!}{(m-1)!(3m-1)!} \sum_{r=1}^{m+1} \frac{(m^2 - mr + (r-1)^2)(m+r-3)!(2m-r-1)}{(r-1)!(m-r+1)!} t^{r-1}.\end{aligned}$$

Using the above relations in (42) and (43) with  $x = 1$ , we obtain the refined enumerations of the half-turn symmetric alternating-sign matrices with 1 and  $-1$  in the centre of a matrix respectively.

## 5.4 Refined 4-enumerations

Let us return again to relations (42) and (43). The denominators of the fractions in the right-hand sides of these relation are  $x - 4$ . From the other hand, the expression in the right-hand side is regular for any  $x$ . Therefore, one should have the equality

$$2 \mathcal{A}(m+1; t, 4) \mathcal{A}_{\text{HT}}^{(2)}(2m; t, 4) = \mathcal{A}(n; t, 4) \mathcal{A}_{\text{HT}}^{(2)}(2m+2; t, 4),$$

which implies the recursive relation

$$2 \frac{\mathcal{A}^2(m+1; t, 4)}{\mathcal{A}^2(m; t, 4)} = \frac{\mathcal{A}_{\text{HT}}(2m+2; t, 4)}{\mathcal{A}_{\text{HT}}(2m; t, 4)}.$$

Using the equalities  $\mathcal{A}_{\text{HT}}(2; t, x) = 1 + t$  and  $\mathcal{A}(1; t, x) = 1$ , one comes to the relation

$$\mathcal{A}_{\text{HT}}(2m; t, 4) = 2^{m-1}(1+t)\mathcal{A}^2(m; t, 4).$$

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